Some characterizations of local bmo and h^{1} on spaces of homogeneous type (Harmonic Analysis and Nonlinear Partial Differential Equations)

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Some characterizations of local bmo and h^1 on spaces of homogeneous type

By

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Abstract

In this paper we generalize some results on the local bmo and Hardy space h^1, shown in [18] for doubling metric-measure spaces, to the setting of spaces of homogeneous type. These include a John-Nirenberg inequality for bmo, proved using a good-lambda inequality as well as by duality, the boundedness of the Hardy-Littlewood maximal function on bmo, and a characterization of h^1 in terms of an atomic decomposition with an approximate moment condition on the atoms, together with the corresponding mean oscillation condition for bmo.

§ 1. Introduction

This article is a follow-up to a previous article by the first and last authors [18] which dealt with the spaces bmo and h^1 in the setting of a metric space with a doubling measure. These spaces, originally defined by Goldberg [20], are "local" versions of the John-Nirenberg space BMO of functions of bounded mean oscillation [22], and the real Hardy space H^1 (see [19]), in the sense that the relevant quantities such as the sharp function and the maximal function, as well as the atomic decomposition, are scale-dependent (the term *nonhomogeneous* is also used to distinguish these from the "global", or homogeneous, versions). Because of the lack of global cancellation requirements and their closedness under multiplication by smooth cut-off functions, Goldberg's local Hardy spaces are better suited for working on domains or on manifolds. In recent years many results on manifolds have been extended to the setting of metric measure spaces. The article [18] generalized to the "local" case on a metric-measure space various properties ranging from the John-Nirenberg inequality and the boundedness of

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the Hardy-Littlewood maximal function for BMO, to duality and the atomic decomposition for H^1.

The purpose of the current paper is to show the modifications required in order to transfer those results to the even more general setting of a space of homogeneous type. This setting is less restrictive since it allows for situations where the topology is locally generated by balls given by a quasi-metric (rather than a metric), which may arise from the underlying geometry or from the study of partial differential equations. In fact, it is the balls and their *engulfing property* that play the crucial role in covering theorems and consequently in much of the harmonic analysis (see [30], [31]). While the study of Hardy spaces on spaces of homogeneous type is not new, as evidenced by the groundbreaking work of Coifman and Weiss [15] and Macias and Segovia [24], [25], it continues to be a topic of interest, especially as regards the minimal assumptions on the measure and the quasi-metric (see [3] for recent work in this area). Our goal here is to indicate what is needed for the results in [18] to hold. This is not just a matter of inserting constants in the triangle inequality. The failure of the triangle inequality has the effect that the balls in the quasi-metric are not necessarily open in the topology they (locally) generate, and the quasi-metric itself is not continuous. Fortunately, as shown by Macias and Segovia [24], when needed one can resort to an equivalent metric which is Hölder continuous, and this is what we do in Section 7 which deals with the Hardy space and the atomic decomposition. For the rest of the material, concerning bmo, such strong assumptions on the metric are not required, but we do need some regularity assumptions on the measure to ensure that the Lebesgue differentiation theorem holds, as well as assumptions on the balls to guarantee that the Hardy-Littlewood maximal function is measurable (note that it may no longer be lower semi-continuous if the balls are not open sets).

The present proceedings article was motivated by a talk given by the third author at the conference Harmonic Analysis and Nonlinear Partial Differential Equations held at RIMS, Kyoto University, Japan in June, 2014, which focused on the results in [18], and therefore much of the following exposition follows closely, in structure and language, that in [18].

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§ 2. Preliminaries

A quasi-metric d on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $\exists \kappa > 0$ such that $d(x, y) \leq \kappa[d(x, z) + d(z, y)]$ for all $x, y, z \in X$ (the *quasi-triangle inequality*).

   If, in addition, d satisfies

3. $d(x, y) = d(y, x)\ \forall x, y \in X$,

then d is called a symmetric quasi-metric. In this paper, we will restrict ourselves to symmetric quasi-metric spaces, as to make discussions of balls meaningful. We call

$$B(x, r) = \{y \in X : d(y, x) < r\}$$

the *open ball* with center x and radius r, which we will also denote by $r(B)$. Note that while, as sets, we may have $B(x, r) = B(x', r')$ for $(x, r) \neq (x', r')$, these will not be considered the same ball, so that when we discuss a ball B in what follows it will always be associated with a particular pair (x, r). We define a set U to be open provided for every $x \in U$, there exists $r > 0$ with $B(x, r) \subset U$. Since the balls centered at each point x are nested, the open sets then form a topology. However, note that when $\kappa > 1$, it is not necessarily true that if $y \in B(x, r)$ then $B(x, r) \supset B(y, r')$ for some $r' > 0$, hence the balls themselves may not be open in this sense.

Taking the Borel sets to be the elements of the σ-algebra generated by the topology, we fix a Borel measure μ, meaning all Borel sets are μ-measurable, and further assume that the balls $B(x, r)$ are also μ-measurable with finite, positive measure for every $x \in X$ and $r > 0$. We also need to assume that μ is Borel regular in the sense that for every μ-measurable set A there exists a Borel set $B \supset A$ with $\mu(A) = \mu(B)$.

For a ball $B = B(x, r)$ and $\delta > 0$ we let δB denote the δ-dilate of B, namely $\delta B = B(x, \delta r)$. We will need two specific dilates in what follows. First, letting $K = \kappa + 2\kappa^2$ (in the metric case $K = 3$), we denote KB by $\widehat{B}$. Note that by the quasi-triangle inequality,

$$B' \cap B \neq \emptyset \text{ and } r(B') \leq r(B) \implies B' \subset \widehat{B}, \tag{2.1}$$

which is known as the *engulfing property* of the balls. Furthermore, since we follow [31] for the Vitali-type covering lemma we use $\widetilde{B}$ to denote the dilate of the ball B by a somewhat larger constant, $\kappa + 4\kappa^2$ (which is 5 in the metric case), but the proof of the lemma can be modified to get it arbitrarily close to K.

An important hypothesis on the measure μ is that it is doubling, i.e. there exists a constant $C_D \geq 1$ such that for all balls B,

$$\mu(2B) \leq C_D \mu(B). \tag{2.2}$$

The constant C_D is called the doubling constant of μ. Corresponding to the dilates of B defined above, let N_1, N_2 be the least integers so that $\mu(\widehat{B}) \leq C_D^{N_1} \mu(B)$ and $\mu(\widetilde{B}) \leq C_D^{N_2} \mu(B)$.

The Hardy-Littlewood maximal function is an essential tool in what follows. For a locally μ-integrable function f on X, we define the average of f over a ball B by

$$f_B := \fint_B f(y) d\mu(y) := \frac{1}{\mu(B)} \int_B f(y) d\mu(y)$$

(recalling that the measure μ of a ball is always finite and positive) and set

$$Mf(x) := \sup_{B \ni x} |f|_B. \tag{2.3}$$

Note this is an *uncentered* maximal function, meaning the supremum is taken over all balls B containing the point x, not necessarily centered at x. Hence if $Mf(x) > \lambda$ then there exists a ball B containing x with $Mf > \lambda$ on B, so the set $E_\lambda = \{x \in X : Mf(x) > \lambda\}$ is a union of balls. However, this does not mean that it is open since the balls themselves are not necessarily open sets, and we are not guaranteed to find a ball *centered* at each point of E_λ which is contained in E_λ. Thus we cannot conclude that Mf is lower semi-continuous. Nevertheless, we must know that E_λ is μ-measurable for every λ in order to guarantee the measurability of the maximal function. Given the measurability of the balls, this will follow, for example, if we assume that any set which is a union of balls is a union of a countable number of balls. This kind of Lindelöf property is automatically true if the balls are open, as can be shown by Whitney-type covering lemmas (see [15], [30]) Another tool that we will need to use, in particular to conclude that $|f| \leq Mf$ almost everywhere, is a version of the Lebesgue differentiation theorem for the balls in the quasi-metric. Such a theorem holds under the assumption, stated above, that μ is a Borel regular measure - see [21], Theorem 1.8, in the case of a metric space. For spaces of homogeneous type various proofs can be found in the literature under certain extra assumptions: Lemma 7 in [13] assumes the density of the continuous functions with compact support, Corollary I.3.1 in [30] assumes certain conditions on the balls, while the proof in [33] claims to suppose only that the balls are themselves spaces of homogeneous type, but in fact what is needed there is the assumption that μ is Borel regular ([16], [4], [3]).

Given a family of balls $\mathcal{B}$, we define a new maximal function relative to this collection by restricting to averages over balls in $\mathcal{B}$:

$$M_{\mathcal{B}}f(x) := \sup\{|f|_B : \ B \in \mathcal{B}, x \in B\}. \tag{2.4}$$

Under the assumption above, this function is also μ-measurable.

Relative to this maximal function we show a Calderón-Zygmund-type decomposition, analogous to that for dyadic cubes in $\mathbb{R}^n$ (see [30], Lemma IV.3.1).

Lemma 2.1. *Suppose $\mathcal{B}$ is a family of balls with radii uniformly bounded above. Let f be a nonnegative function in $L^1_{\mathrm{loc}}(X)$. Fix $\lambda > 0$ and let $E_\lambda = \{x \in X : M_{\mathcal{B}}f(x) > \lambda\}$. Then there exists a sequence of balls $\{B_k\} \subset \mathcal{B}$ such that*

(i) *the balls B_k are pairwise disjoint and $\displaystyle\bigcup_{k=1}^{\infty} B_k \subset E_\lambda \subset \bigcup_{k=1}^{\infty} \widetilde{B_k}$;*

(ii) $\fint_{B_k} f(x)dx > \lambda$ *for every k; and*

(iii) $\fint_{\widetilde{B_k}} f(x)dx \leq \lambda$ *whenever $\widetilde{B_k} \in \mathcal{B}$.*

These conclusions do not require (2.2). If (2.2) holds, we have, in addition, that

$$\mu(E_\lambda) \leq \frac{C_D^{N_2}}{\lambda} \int_{E_\lambda} f(x)d\mu. \tag{2.5}$$

Moreover, if the family of balls $\mathcal{B}$ also satisfies the condition that for each $x \in X$, there exists some $\epsilon > 0$ with $B(x,r) \in \mathcal{B}$ for all $r < \epsilon$, then

(iv) $f(x) \leq \lambda$ *for a.e. $x \in X \setminus E_\lambda$.*

Proof. The proof is the same as in [18] with minor modifications in the current setting. In particular, for each $x \in E_\lambda$, we take a ball $B = B_x$ containing x such that

$$B \in \mathcal{B} \ \text{ and} \fint_B f d\mu > \lambda. \tag{2.6}$$

If (2.6) holds with $B = \widetilde{B_x}$, we replace B_x by $\widetilde{B_x}$, and repeat until this fails, namely either $\widetilde{B_x} \notin \mathcal{B}$, or $\widetilde{B_x} \in \mathcal{B}$ but

$$\fint_{\widetilde{B_x}} f d\mu \leq \lambda.$$

Since the radii of the balls is bounded above, this process terminates and the final choice of the ball B_x satisfies conditions (ii) and (iii). Moreover, by (2.6), $B_x \subset E_\lambda$, hence $\bigcup_{x \in E_\lambda} B_x = E_\lambda$. We apply the Vitali-type covering lemma, Lemma 3.3, in [31] to the collection $\{B_x\}_{x \in E_\lambda}$ (note that their proof only uses the engulfing property (2.1) and the fact that the balls B_x have radii uniformly bounded above), to get a countable subcollection $\{B_k\}_{k\geq 1}$ of pairwise disjoint balls for which (i) holds.

The inequality (2.5) follows from (i), (ii), and additivity in the same way as for the metric case, with the constants adjusted to the current setting (see the definition of N_1 and N_2 following (2.2)).

Property (iv) follows from the bound $f \leq M_{\mathcal{B}} f \leq \lambda$ almost everywhere on the set $X \setminus E_\lambda$, which is a consequence of the Lebesgue differentiation theorem for the balls in the quasi-metric (see the discussion preceding (2.4)). □

Remark. *For a locally integrable function f, we can get the weak-type inequality for the unrestricted maximal function:*

$$\mu(\{x \in X : Mf(x) > \lambda\}) \leq \frac{C_D^{N_2}}{\lambda} \int_X |f(x)| d\mu \tag{2.7}$$

from Lemma 2.1, as in the proof of the maximal theorem (Theorem 2.2) in [21], by considering the maximal function $M_R := M_{\mathcal{B}}$ taken over the collection of balls $\mathcal{B} = \{B : r(B) < R\}$, for some $R < \infty$, and then taking the limit on the left-hand-side of (2.5) as $R \to \infty$, noting that

$$\{x \in X : Mf(x) > \lambda\} = \{x \in X : \sup_R M_R f(x) > \lambda\} = \bigcup_R \{x \in X : M_R f(x) > \lambda\}.$$

§ 3. The space bmo(X) and the John-Nirenberg inequality

The following definition of the local space of functions of bounded mean oscillations, which is identical to that in [18] for the case of a metric-measure space, is a slight variation of the original definition in [20] where $R = 1$:

Definition 3.1. Fix $R > 0$. For each ball B, let

$$c_B = \begin{cases} f_B := \fint_B f & \text{if } r(B) < R, \\ 0 & \text{if } r(B) \geq R. \end{cases} \tag{3.1}$$

With c_B as in (3.1) and $f \in L^1_{\text{loc}}(X)$, define, for $x \in X$,

$$f^*(x) = \sup_{B \ni x} \fint_B |f - c_B| d\mu. \tag{3.2}$$

If $f^* \in L^\infty(X)$ we say that f is in bmo(X), and write

$$\|f\|_{\text{bmo}} := \|f^*\|_{L^\infty}. \tag{3.3}$$

When $R = \infty$, or when $R > \text{diam}(X) := \sup\{d(x, y) : x, y \in X\}$ and we assume that every ball B has radius $r(B) < R$, the function f^* reverts to the usual sharp function

$$f^\#(x) = \sup_{x \in B} \fint_B |f - f_B| d\mu, \quad \forall x \in X, \tag{3.4}$$

and bmo(X) is the space of functions of bounded mean oscillation BMO(X), in which (3.3) defines a norm $\|f\|_{\text{BMO}}$ modulo constant functions.

From here onward we fix $R < \mathrm{diam}(X)$, unless otherwise stated, so that there exist balls with $r(B) \geq R$, forcing nonzero constant f to have $\|f\|_{\mathrm{bmo}} \neq 0$. In this case $\|f\|_{\mathrm{bmo}}$ defines a norm, which depends on the choice of the constant R. However, since

$$\fint_B |f(x) - f_B| d\mu \leq 2 \fint_B |f(x)| d\mu, \tag{3.5}$$

if f satisfies the definition with $R = R_0$, it will satisfy it for all $R > R_0$. In Sections 6 and 7 we will connect the choice of R with the definition of the local Hardy space $h^1(X)$ and show that both $\mathrm{bmo}(X)$ and $h^1(X)$ are independent of this choice.

Finally, note that (3.5) gives $f^{\#} \leq 2 f^*_{\mathcal{B}}$, hence $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{\mathrm{bmo}}$. Trivially, we also have $\|f\|_{\mathrm{bmo}} \leq 2\|f\|_\infty$, resulting in the set inclusions $L^\infty(X) \subset \mathrm{bmo}(X) \subset \mathrm{BMO}(X)$. It is useful to note (see [18]) that if $f \in \mathrm{bmo}$ then $|f| \in \mathrm{bmo}$ with

$$\||f|\|_{\mathrm{bmo}} \leq 2\|f\|_{\mathrm{bmo}}. \tag{3.6}$$

The following version of the John-Nirenberg inequality is identical to that in [18]. As pointed out there, inequality (3.8), the John-Nirenberg inequality for $\mathrm{BMO}(X)$, gives the "local" inequality (3.7) for a function $f \in \mathrm{bmo}(X)$ in the case of balls with $r(B) < R$, with c replaced by $c/2$, since $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{\mathrm{bmo}}$. For a doubling metric-measure space, the case of (3.7) for balls with $r(B) \geq R$ can be proved similarly to the proof of (3.8), the John-Nirenberg inequality for $\mathrm{BMO}(X)$, found in Section 5 of [1], which in turn is based on the proof given in [29] for the Euclidean case and attributed to unpublished work of Calderón. Here, as in [18], we give two alternative proofs for both the "local" and "global" inequalities, one based on a good-λ inequality (Section 4) and the other based on duality (Section 7).

Theorem 3.2. *There exist two positive constants C and c (depending only on the doubling constant C_D and the quasi-triangle inequality constant κ) such that, given a function f in* $\mathrm{bmo}(X)$*, for any ball B, taking , $k = c\|f\|^{-1}_{\mathrm{bmo}}$, and c_B is as in (3.1), we have*

$$\mu(\{x \in B : |f(x) - c_B| > \lambda\}) \leq C\mu(B)\exp(-k\lambda) \quad \forall \lambda > 0. \tag{3.7}$$

Moreover, for $f \in \mathrm{BMO}(X)$, the inequality

$$\mu(\{x \in B : |f(x) - f_B| > \lambda\}) \leq C\mu(B)\exp(-k'\lambda) \quad \forall \lambda > 0 \tag{3.8}$$

holds for all balls B with $k' = c\|f\|^{-1}_{\mathrm{BMO}}$.

Conversely, if (3.7) holds for some positive constants C and k, for all balls B, then $f \in \mathrm{bmo}(X)$.

This last statement is a consequence of the identity

$$\frac{1}{\mu(B)} \int_B |f(x) - c_B|^p dx = \frac{p}{\mu(B)} \int_0^\infty \lambda^{p-1} \mu(\{x \in B : |f(x) - c_B| > \lambda\}) d\lambda$$

for $p = 1$, which, together with Hölder's inequality, also gives the following equivalence of p norms on bmo:

Corollary 3.3. *For $1 \leq p < \infty$, let*

$$\|f\|_{\mathrm{bmop}} := \sup_B \left[\fint_B |f(x) - c_B|^p dx \right]^{1/p}, \tag{3.9}$$

where the supremum is taken over all balls B, and c_B is as in (3.1). Then

$$\|f\|_{\mathrm{bmo}} \leq \|f\|_{\mathrm{bmop}} \leq A\|f\|_{\mathrm{bmo}},$$

where $A = [pC \cdot \Gamma(p)/c]^{1/p}$ for C, c, the constants in Theorem 3.2, and $\Gamma(y) := \int_0^\infty u^{y-1}e^{-u}du$.

§ 4. The John-Nirenberg inequality via a good-λ inequality

As in [18], we start with the following variants of definitions (3.1) and (3.2) corresponding to a family of balls $\mathcal{B}$ and another family, $\widetilde{\mathcal{B}}$ (to be specified below), consisting of balls which are not in $\mathcal{B}$:

$$c_B = \begin{cases} f_B & \text{if } B \in \mathcal{B}, \\ 0 & \text{if } B \in \widetilde{\mathcal{B}}. \end{cases} \tag{4.1}$$

and

$$f^*_{\mathcal{B}}(x) = \sup_{x \in B \in \mathcal{B} \cup \widetilde{\mathcal{B}}} \fint_B |f - c_B| d\mu, \quad \forall x \in X. \tag{4.2}$$

When $\mathcal{B} = \{B : r(B) < R\}$ and $\widetilde{\mathcal{B}} = \{B : r(B) \geq R\}$, we recover f^*.

The following relation between $f^*_{\mathcal{B}}$ and the maximal function $M_{\mathcal{B}}f$ is called a good-λ inequality (also known as a *relative distributional inequality* - see [30] , Section IV.3.6 for the case of the dyadic maximal and sharp functions corresponding to $\mathrm{BMO}(\mathbb{R}^n)$).

Lemma 4.1. *Let $f \in L^1_{\mathrm{loc}}(X)$. Consider a collection of balls $\mathcal{B}$ with radii uniformly bounded above, and define the corresponding collection of balls $\widetilde{\mathcal{B}}$ by*

$$\widetilde{\mathcal{B}} = \{\widetilde{B} : \widetilde{B} \notin \mathcal{B}, B \in \mathcal{B}\}. \tag{4.3}$$

Given two constants $0 < b < 1$ and $c > 0$, for all $\lambda > 0$,

$$\mu(\{x : M_{\mathcal{B}}f(x) > \lambda, f^*_{\mathcal{B}}(x) \leq c\lambda\}) \leq a\mu(\{x : M_{\mathcal{B}}f(x) > b\lambda\}), \tag{4.4}$$

where $a = \frac{C_D^{2N_2} c}{(1-b)}$.

Proof. As in [18], we prove the inequality for $f \geq 0$ since the general case for an integrable function follows, with the constant a corresponding to $2c$ instead of c, from the fact that $M_{\mathcal{B}}(f) = M_{\mathcal{B}}(|f|)$ and $|f|^*_{\mathcal{B}}(x) \leq 2f^*_{\mathcal{B}}(x)$ by (3.6).

Fix $\lambda > 0$. As explained in [18], we can use a slight modification of the proof of Lemma 2.1 to get a covering of the set $E_{b\lambda} := \{x \in X : M_{\mathcal{B}}f(x) > b\lambda\}$ by $\bigcup_{i\geq 1} \widetilde{B_i}$, where the pairwise disjoint sequence of balls $\{B_i\}$ satisfies properties (i)-(iii) with respect to $b\lambda$, and in addition, since $E_\lambda \subset E_{b\lambda}$ (as $b < 1$), every $x \in E_\lambda$ lies in a ball B, satisfying (2.6), with $B \subset \widetilde{B_i}$ for some i, and therefore $M(f\chi_{\widetilde{B_i}})(x) > \lambda$.

Now, for each i, denote by E^i_λ the set of $x \in E_\lambda$ for which the statement above holds, so that $E_\lambda = \cup E^i_\lambda$. We want to estimate the left-hand-side of (4.4) by writing

$$\mu(\{x \in E_\lambda : f^*_{\mathcal{B}}(x) \leq c\lambda\}) \leq \sum_i \mu(\{x \in E^i_\lambda : f^*_{\mathcal{B}}(x) \leq c\lambda\}). \tag{4.5}$$

We first use the the weak inequality (2.7), replacing f by $f\chi_{\widetilde{B_i}}$, to obtain

$$\mu(E^i_\lambda) \subset \mu(\{x : M(f\chi_{\widetilde{B_i}})(x) > \lambda\}) \le \frac{C_D^{N_2}}{\lambda}\int_X |f\chi_{\widetilde{B_i}}|d\mu = \frac{C_D^{N_2}}{\lambda}\int_{\widetilde{B_i}} |f|d\mu. \tag{4.6}$$

This estimate will take care of the case $\widetilde{B_i} \notin \mathcal{B}$, since in that case $B_i \in \mathcal{B}$ implies $\widetilde{B_i} \in \widetilde{\mathcal{B}}$ and $c_{\widetilde{B_i}} = 0$ by (4.1).

If $\widetilde{B_i} \in \mathcal{B}$ (so $c_{\widetilde{B_i}} = f_{\widetilde{B_i}}$) then property (iii) in Lemma 2.1 gives $f_{\widetilde{B_i}} \le b\lambda$, hence

$$E^i_\lambda \subset \{x : M[(f - f_{\widetilde{B_i}})\chi_{\widetilde{B_i}}](x) > (1-b)\lambda\}, \tag{4.7}$$

and we can apply (2.7) to $|f - f_{\widetilde{B_i}}|\chi_{\widetilde{B_i}}$ to get

$$\mu(\{x : M[(f - f_{\widetilde{B_i}})\chi_{\widetilde{B_i}}](x) > (1-b)\lambda\}) \le \frac{C_D^{N_2}}{(1-b)\lambda}\int_{\widetilde{B_i}} |f - f_{\widetilde{B_i}}|d\mu. \tag{4.8}$$

Combining the estimates (4.6) - (4.8) in the two cases above with the definition of the sharp function $f^*_{\mathcal{B}}$, and noting that $0 < b < 1$, we have

$$\begin{aligned}
\mu(\{x \in E^i_\lambda : f^*_{\mathcal{B}}(x) \le c\lambda\}) \le \mu(E^i_\lambda) &\le \frac{C_D^{N_2}}{(1-b)\lambda}\int_{\widetilde{B_i}} |f - c_{\widetilde{B_i}}|d\mu \\
&\le \frac{C_D^{N_2}\mu(\widetilde{B_i})}{(1-b)\lambda}\inf_{x\in\widetilde{B_i}} f^*_{\mathcal{B}}(x) \\
&\le \frac{C_D^{N_2}\mu(\widetilde{B_i})}{(1-b)\lambda}\, c\lambda \\
&\le \frac{C_D^{2N}c}{(1-b)}\mu(B_i). \qquad (4.9)
\end{aligned}$$

For the third estimate we assumed $E^i_\lambda \cap \{x : f^*_{\mathcal{B}}(x) \le c\lambda\} \ne \emptyset$, otherwise the inequality holds trivially.

Now sum (4.9) over i, using the fact that the B_i are disjoint subsets of $E_{b\lambda}$ (property (i) of Lemma 2.1) and the additivity of μ, and combine with (4.5) to obtain (4.4), completing the proof of Lemma 4.1. □

Again following [18], we will now prove the John-Nirenberg inequality for bmo (Theorem 3.2) as a corollary of Lemma 4.1. The proof is analogous to that in [30], Section IV.3.7 for BMO($\mathbb{R}^n$), while for BMO on a space of homogeneous type,a similar technique is used in [26].

Proof of Theorem 3.2. Let B_0 be a given ball, set $R_0 := r(B_0)$, the radius of B_0, and consider the collection of smaller balls which intersect B_0:

$$\mathcal{B} = \{B : B \cap B_0 \ne \emptyset, r(B) < R_0\}.$$

By the engulfing property (2.1) all balls in $\mathcal{B}$ are contained in $\widehat{B_0}$, hence the maximal function $M_{\mathcal{B}}f$ vanishes outside $\widehat{B_0}$, $E_\lambda := \{x \in X : M_{\mathcal{B}}f(x) > \lambda\} \subset \widehat{B_0}$, and

$$\mu(E_\lambda) \le C_D^{N_1}\mu(B_0) \text{ for all } \lambda > 0. \tag{4.10}$$

Take $\widetilde{\mathcal{B}}$ as in (4.3); if $\widetilde{B} \in \widetilde{\mathcal{B}}$, namely $\widetilde{B} \notin \mathcal{B}$ but $B \in \mathcal{B}$, then we must have that $\widetilde{B} \cap B_0 \neq \emptyset$, hence $r(\widetilde{B}) \geq R_0$ (in fact $R_0 \leq r(\widetilde{B}) < (\kappa + 4\kappa^2)R_0$). Thus

$$f^*_{\mathcal{B}}(x) \leq \max\left(\sup_{x\in B, r(B)<R_0} \fint_B |f - f_B| d\mu, \sup_{x\in B, r(B)\geq R_0} \fint_B |f| d\mu \right). \tag{4.11}$$

Consider the following two cases (not mutually exclusive):

Case 1 (large ball/bmo): We assume $f \in \mathrm{bmo}(X)$ and $R_0 \geq R$. Set $g = |f|$, $\gamma = 2\|f\|_{\mathrm{bmo}}$. Applying (4.11) to g and using (3.6), then (3.5), we have

$$\begin{aligned}
g^*_{\mathcal{B}}(x) &\leq \max\left(\sup_{x\in B, r(B)<R_0} \fint_B ||f| - |f|_B| d\mu, \sup_{x\in B, r(B)\geq R_0} \fint_B |f| d\mu \right) \\
&\leq \max\left(2 \sup_{x\in B, r(B)<R_0} \fint_B |f - f_B| d\mu, \sup_{x\in B, r(B)\geq R_0} \fint_B |f| d\mu \right) \\
&\leq 2\max\left(\sup_{x\in B, r(B)<R} \fint_B |f - f_B| d\mu, \sup_{x\in B, r(B)\geq R} \fint_B |f| d\mu \right) \\
&= \gamma.
\end{aligned}$$

Case 2 (small ball/BMO): Assume $f \in \mathrm{BMO}(X)$ and set $g = |f_1|$, where $f_1 = (f - f_{B_0})\chi_{\widehat{B_0}}$, and $\gamma = 2C_D^{3N_1}\|f\|_{\mathrm{BMO}}$. Here we will not distinguish between $R_0 < R$ or $R_0 \geq R$, so that the proof applies to any ball in the case $f \in \mathrm{BMO}(X)$. If in addition $f \in \mathrm{bmo}(X)$ then, as previously mentioned, this case will give us (3.7) for small balls ($R_0 < R$), since $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{\mathrm{bmo}}$.

As previously noted, all the balls in $\mathcal{B}$ are contained in $\widehat{B_0}$, so we have that $M_{\mathcal{B}} f_1$ is supported in $\widehat{B_0}$ and

$$\sup_{x\in B\in\mathcal{B}} \fint_B |f_1 - (f_1)_B| d\mu = \sup_{x\in B\in\mathcal{B}} \fint_B |f - f_B| d\mu \ \leq \|f\|_{\mathrm{BMO}}. \tag{4.12}$$

Moreover, if $\widetilde{B} \in \widetilde{\mathcal{B}}$, then $\widetilde{B} \cap B_0 \neq \emptyset$ and $r(\widetilde{B}) \geq R_0$, which, by the engulfing property, gives $\widehat{\widetilde{B}} \supset B_0$, hence $\mu(\widetilde{B}) \geq C_D^{-N_1}\mu(B_0) \geq C_D^{-2N_1}\mu(\widehat{B_0})$ and

$$\begin{aligned}
\fint_{\widetilde{B}} |f_1| d\mu &= \frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}\cap\widehat{B_0}} |f - f_{B_0}| d\mu \leq C_D^{2N_1} \fint_{\widehat{B_0}} |f - f_{B_0}| d\mu \\
&\leq C_D^{2N_1} \left\{ \fint_{\widehat{B_0}} |f - f_{\widehat{B_0}}| d\mu + |f_{B_0} - f_{\widehat{B_0}}| \right\} \\
&\leq C_D^{2N_1} \left\{ \fint_{\widehat{B_0}} |f - f_{\widehat{B_0}}| d\mu + C_D^{N_1} \fint_{\widehat{B_0}} |f - f_{\widehat{B_0}}| d\mu \right\} \\
&\leq 2C_D^{3N_1}\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

Taking the supremum over all $\widetilde{B} \in \widetilde{\mathcal{B}}$ and combining with (4.12) and (3.6), we see that

$$g^*_{\mathcal{B}}(x) = |f_1|^*_{\mathcal{B}} \leq \max(2\|f\|_{\mathrm{BMO}}, 2C_D^{3N_1}\|f\|_{\mathrm{BMO}}) = \gamma.$$

Thus in both cases we have shown, for the respective choices of g and γ, that

$$g^*_{\mathcal{B}} \leq \gamma. \tag{4.13}$$

Continuing with either of those choices, let E_λ denote the set $\{x \in X : M_{\mathcal{B}}g(x) > \lambda\}$, and note that (4.10) still holds in both cases. Moreover, since the collection of balls $\mathcal{B}$ contains balls of arbitrarily small radius centered at each $x \in B_0$, we can apply the Lebesgue Differentiation Theorem (see the discussion preceding (2.4)) to conclude $g \leq M_{\mathcal{B}}g$ a.e. on B_0, hence

$$\mu(\{x \in B_0 : g(x) > \lambda\}) \leq \mu(E_\lambda). \tag{4.14}$$

The John-Nirenberg inequalities in both cases are thus reduced to estimating $\mu(E_\lambda)$.

In order to use the good-λ inequality, we take advantage of (4.13) (assuming, of course, that $\gamma \neq 0$, since otherwise our function is zero or constant) and put $c = \gamma/\lambda$ in (4.4), so that $g^*_{\mathcal{B}} \leq \gamma = c\lambda$, and hence

$$\{x : M_{\mathcal{B}}g(x) > \lambda, g^*_{\mathcal{B}}(x) \leq c\lambda\} = E_\lambda.$$

Applying (4.4) to g with $0 < b < 1$ gives

$$\mu(E_\lambda) \leq a\mu(E_{b\lambda}), \quad a = \frac{C_D^{2N_2}c}{1-b} = \frac{C_D^{2N_2}\gamma}{\lambda(1-b)}.$$

Set $\lambda_0 := 2C_D^{2N_2}\gamma$. For $\lambda > \lambda_0$, putting $b = 1 - \lambda_0/\lambda$, we get $0 < b < 1$, $b\lambda = \lambda - \lambda_0$, and $a = \frac{1}{2}$, giving

$$\mu(E_\lambda) \leq \frac{1}{2}\mu(E_{\lambda-\lambda_0}). \tag{4.15}$$

We will iterate this inequality k times, where k is the largest integer in $\frac{\lambda-\lambda_0}{\lambda_0}$. More specifically, if we apply (4.15) to λ replaced by $\lambda_j = (j+1)\lambda_0$, $j = k, \ldots, 1$, then

$$\mu(E_\lambda) \leq \mu(E_{\lambda_k}) \leq \left(\frac{1}{2}\right)^k \mu(E_{\lambda_0}) \leq \left(\frac{1}{2}\right)^{\frac{\lambda}{\lambda_0}-2} C_D^{N_1}\mu(B_0) = C_1 e^{-c_1\lambda/\gamma}\mu(B_0), \tag{4.16}$$

where $C_1 = 4C_D^{N_1}$, $c_1 = (2C_D^{2N_2})^{-1}\log 2$, and we have used (4.10).

When $\lambda \leq \lambda_0$, we have trivially from (4.10) that $\mu(E_\lambda) \leq C_2 e^{-c_2\lambda/\gamma}\mu(B_0)$ with $C_2 = C_D^{N_1}e$ and $c_2 = (2C_D^{2N_2})^{-1}$. Combining this, (4.16) and (4.14), we see that

$$\mu(\{x \in B_0 : g(x) > \lambda\}) \leq C_3 e^{-c_3\lambda/\gamma}\mu(B_0)$$

for all $\lambda > 0$, with $C_3 = \max(C_1, C_2) = 4C_D^{N_1}$ and $c_3 = \min(c_1, c_2) = (2C_D^{2N_2})^{-1}\log 2$.

Letting $C = C_3$, $c = c_3/2$ (not to be confused with the constant c in the good-λ inequality (4.4)) and substituting $g = |f|$, $f \in \mathrm{bmo}(X)$ and $\gamma = 2\|f\|_{\mathrm{bmo}}$ as in Case 1, we get (3.7) for B_0 large ($c_{B_0} = 0$). On the other hand, putting $C = C_3$, $c = (4C_D^{3N_1})^{-1}c_3$, $g = |f_1| = |f - f_{B_0}|\chi_{\widehat{B_0}}$ and $\gamma = 2C_D^{3N_1}\|f\|_{\mathrm{BMO}}$ as in Case 2 gives (3.8) for $f \in \mathrm{BMO}(X)$. When $f \in \mathrm{bmo}(X)$ and B_0 is small ($c_{B_0} = f_{B_0}$), these choices in turn give (3.7), noting that $\gamma \leq 4C_D^{3N_1}\|f\|_{\mathrm{bmo}}$. □

The second corollary of Lemma 4.1 is the following:

Corollary 4.2. *For $1 \leq p < \infty$,*

$$\|M_{\mathcal{B}}f\|_p \leq C_p\|f^*_{\mathcal{B}}\|_p,$$

provided $M_{\mathcal{B}}f \in L^{p_0}$ for some $p_0 \leq p$.

The proof (again it suffices to consider $f \geq 0$) follows from Lemma 4.1 by writing

$$
\begin{aligned}
\mu(\{x : M_{\mathcal{B}}f(x) > \lambda\}) &\leq \mu(\{x : M_{\mathcal{B}}f(x) > \lambda, f^*_{\mathcal{B}}(x) \leq c\lambda\}) + \mu(\{x : f^*_{\mathcal{B}}(x) > c\lambda\}) \\
&\leq a\mu(\{x : M_{\mathcal{B}}f(x) > b\lambda\}) + \mu(\{x : f^*_{\mathcal{B}}(x) > c\lambda\}),
\end{aligned}
\tag{4.17}
$$

and integrating in λ against λ^{p-1}, as in the proof of Lemma 2 in Section 3.5, Chapter IV of [30] (see also the remark following the proof for the case $p_0 < p$). In order to subtract the first term obtained on the right, $ab^{-p}\|M_{\mathcal{B}}f\|_p$, from the left-hand-side, we need to assume it is finite and choose $b \in (0,1)$ and c sufficiently small so that $a = \frac{C_D^{2N}c}{(1-b)} \leq b^p$.

Now take $M_{\mathcal{B}} = M_R$ (corresponding to the collection of all balls with radii smaller than R) and denote by f^*_R the sharp function f^* defined in (3.2), to indicate the constant used in (3.1). We also need to define the corresponding maximal function for "large" balls,

$$
M^R f(x) = \sup \{|f|_B : \ x \in B, \ r(B) \geq R\} . \tag{4.18}
$$

Note that if $f^\#$ is the usual (BMO) sharp function defined in (3.4), then

$$
f^*_R \leq \max(f^\#, M^R f).
$$

Thus we can deduce the following inequality from (4.17):

$$
\begin{aligned}
\mu(\{x : M_R f(x) > \lambda\}) \leq a\mu(\{x : M_R f(x) > b\lambda\}) \\
+\mu(\{x : f^\#(x) > c\lambda\}) + \mu(\{x : M^R f(x) > c\lambda\}).
\end{aligned}
$$

Let us fix λ and consider this as $R \to \infty$. Note that $M^R f(x)$ is decreasing in R and converges to zero for every x, provided $f \in L^{p_0}$ for some $p_0 \geq 1$ and $\mu(B) \to \infty$ uniformly as $r(B) \to \infty$, since this guarantees

$$
\sup_{r(B)\geq R} \fint_B |f| d\mu \leq \sup_{r(B)\geq R} \left(\fint_B |f|^{p_0} d\mu\right)^{1/p_0} \leq \sup_{r(B)\geq R} \frac{\|f\|_{p_0}}{\mu(B)^{1/p_0}} \ \to \ 0 \quad \text{as } R \to \infty.
$$

Thus the sets $\{x : M^R f(x) > c\lambda\}$ shrink to $\emptyset$. Moreover, the boundedness of the maximal function on L^{p_0} gives us that Mf, and hence M^R, belongs to L^{p_0} (or weak-L^1 if $p_0 = 1$), so the measures $\mu(\{x : M^R f(x) > c\lambda\})$ are finite, hence decay to zero as $R \to \infty$. As noted in Remark 2, when $R \to \infty$, $\mu(\{x : M_R f(x) > \lambda\})$ tends to $\mu(\{x : Mf(x) > \lambda\})$ for every λ, so in the limit we get

$$
\mu(\{x : Mf(x) > \lambda\}) \leq a\mu(\{x : Mf(x) > b\lambda\}) + \mu(\{x : f^\#(x) > c\lambda\}).
$$

But this is just (4.17) for the usual Hardy-Littlewood maximal function and sharp function, so the same proof gives us the following result, proved by Fefferman and Stein in the Euclidean case ([19], Theorem 5):

Corollary 4.3. *Assume that the measure μ satisfies the condition that $\mu(B) \to \infty$ uniformly as $r(B) \to \infty$. Then for $1 < p < \infty$, if $f^\# \in L^p$ and $f \in L^{p_0}$ for some $p_0 \in [1, p]$, we have*

$$
\|Mf\|_p \leq C_p \|f^\#\|_p. \tag{4.19}
$$

The example of a constant function shows that (4.19) may fail even though Corollary 4.2 holds with $M_{\mathcal{B}} = M_R$ for all $R > 0$, so the hypotheses are necessary.

§ 5. Boundedness of the maximal function

In the previous section we considered the L^p boundedness of the Hardy-Littlewood maximal function in relation to the sharp function. When $p = \infty$, we will now show that if $f^* \in L^\infty$, i.e. $f \in \mathrm{bmo}(X)$, then Mf is also in $\mathrm{bmo}(X)$. The corresponding result for BMO, proved in the Euclidean case by Bennett, DeVore and Sharpley (Theorem 4.2 in [6]), is that if $f \in \mathrm{BMO}$ then $Mf \in \mathrm{BMO}$, provided Mf is not identically infinite. In particular, one can write (see [30], Chapter IV, Section 6.3(c))

$$\|Mf\|_{\mathrm{BMO}} \leq c(\|f\|_{\mathrm{BMO}} + |f_{B(0,1)}|),$$

so that for $f \in \mathrm{bmo}(\mathbb{R}^n)$ (with $R = 1$) we get $\|Mf\|_{\mathrm{BMO}} \leq 3c\|f\|_{\mathrm{bmo}}$. We will improve this by having the bmo norm on the left-hand-side. For other proofs of the boundedness of the maximal function on BMO, as well as in the metric-measure setting, we refer to [2], [5], and [7].

Theorem 5.1. *The Hardy-Littlewood maximal operator M is bounded from* $\mathrm{bmo}(X)$ *into* $\mathrm{bmo}(X)$*, i.e. there exists a constant C such that*

$$\|Mf\|_{\mathrm{bmo}} \leq C\|f\|_{\mathrm{bmo}}. \tag{5.1}$$

Proof. The following is a slight modification of the proof given in [18], which was an adaptation of the proof of the corresponding result for $\mathrm{BMO}(\mathbb{R}^n)$, Theorem 4.2 in [6]. However, we will show that for f in $\mathrm{bmo}(X)$, the the maximal function Mf is locally integrable, which is different from the situation in BMO.

Since $M(|f|) = Mf$ and $\| |f| \|_{\mathrm{bmo}} \leq 2\|f\|_{\mathrm{bmo}}$, we only need to consider the case $f \geq 0$. For $x \in X$, denote $Mf(x)$ by $F(x)$. In order to consider averages of F over balls, we need to know $F \in L^1_{\mathrm{loc}}(X)$.

As above, we will use M_R to denote the restricted maximal function (2.4), where the collection $\mathcal{B}$ consists "small" balls relative to some fixed radius R, and M^R to denote the corresponding maximal function for "large" balls, as in (4.18). Then

$$Mf(x) = \max\left\{M_R f(x),\ M^R f(x)\right\}.$$

Taking the same R as in Definition 3.1, for convenience of notation we set $F_1 = M_R f$, $F_2 = M^R f$. We get immediately that F_2 is bounded, namely

$$F_2(x) \leq \|f\|_{\mathrm{bmo}}\ \forall x \in X. \tag{5.2}$$

As for F_1, we fix a ball B_0 and recall that if $B \cap B_0 \neq \emptyset$ and $r(B) < R$ then $B \subset B_0'$, where B_0' is a ball with the same center as B_0 and radius at least $\kappa(r(B_0) + 2\kappa R)$. Therefore

$$F_1(x) \leq M\left(f\chi_{B_0'}\right)(x), \quad x \in B_0. \tag{5.3}$$

By Corollary 3.3, we know $f \in L^p_{\mathrm{loc}}(X)$ for all $p \in (1, \infty)$ and therefore by the L^p boundedness of the maximal function, $M\left(f\chi_{B_0'}\right) \in L^p(X)$. This shows $F_1 \in L^p(B_0)$. In particular, combining this with (5.2), we have shown that F is locally integrable on X.

Now fix the ball B_0 and consider the following cases:
Case 1, $r(B_0) \geq R$: Here we want to show

$$\fint_{B_0} F d\mu \leq C\|f\|_{\mathrm{bmo}}. \tag{5.4}$$

From (5.2), it suffices to show the estimate for F_1.

Applying (5.3) with $B_0' = \widehat{B_0}$ (since in this case $r(\widehat{B_0}) = (\kappa+2\kappa^2)r(B_0) \geq \kappa(r(B_0)+2\kappa R)$), followed by Cauchy-Schwarz, the boundedness of the maximal function on L^2 and Corollary 3.3 with $p = 2$ (noting that $C_{\widehat{B_0}} = 0$), we have

$$\int_{B_0} F_1 d\mu \leq \|M(f\chi_{\widehat{B_0}})\|_2\, \mu(B_0)^{1/2} \leq A_2 \left(\fint_{\widehat{B_0}} |f|^2 d\mu\right)^{1/2} \mu(\widehat{B_0})^{1/2}\mu(B_0)^{1/2} \leq A'\|f\|_{\mathrm{bmo}}\mu(B_0),$$

where A' is a constant depending on A_2, C_D and κ. The inequality (5.4) now follows.
Case 2, $r(B_0) < R$: In this case the desired estimate is

$$\fint_{B_0} |F(x) - F_{B_0}| d\mu \leq C\|f\|_{\mathrm{bmo}}. \tag{5.5}$$

The proof of this estimate, below, following the ideas contained in [6], is equivalent to showing boundedness of the maximal function on $\mathrm{BMO}(X)$.

We again divide the maximal function into two pieces, this time relative to the radius of the ball B_0; set $r = r(B_0)$,

$$M_1 f(x) := M_r f,\ M_2 f(x) := M^r f.$$

Manipulating the integral on the left-hand-side of (5.5) as in the proof of Theorem 4.2 in [6], namely noting that $F = \max(M_1, M_2)$ and therefore

$$\fint_{B_0} |F(x) - F_{B_0}| = \frac{2}{\mu(B_0)} \int_{\{x:F(x)>F_{B_0}\}} [F(x) - F_{B_0}] = \frac{2}{\mu(B_0)} \sum_{i=1}^{2} \int_{\Omega_i} [M_i f(x) - F_{B_0}],$$

where $\Omega_1 = \{x \in B_0 : F_{B_0} < F(x) = M_1 f(x)\}$, $\Omega_2 = \{x \in B_0 : F(x) > F_{B_0}, F(x) > M_1 f(x)\}$, we are reduced to showing that

$$\int_{\Omega_i} [M_i f(x) - F_{B_0}] d\mu \leq C\mu(B_0)\|f\|_{\mathrm{bmo}}, \quad i = 1, 2. \tag{5.6}$$

For $i = 1$, again applying (5.3) with $B_0' = \widehat{B_0} := (\kappa + 2\kappa^2)B_0$ so that $r(B_0') = (\kappa + 2\kappa^2)r(B_0) \geq \kappa(r(B_0) + 2\kappa r)$, we have that on B_0,

$$M_1 f \leq M(f\chi_{\widehat{B_0}}) \leq M[(f - f_{\widehat{B_0}})\chi_{\widehat{B_0}}] + M[f_{\widehat{B_0}}\chi_{\widehat{B_0}}].$$

Again by the L^2 boundedness of the maximal function and Corollary 3.3 with $p = 2$, we have

$$\begin{aligned} \|M[(f - f_{\widehat{B_0}})\chi_{\widehat{B_0}}]\|_2 &\leq A_2\|(f - f_{\widehat{B_0}})\chi_{\widehat{B_0}}\|_2 = A_2 \left(\mu(\widehat{B_0})\fint_{\widehat{B_0}} |f - f_{\widehat{B_0}}|^2 d\mu\right)^{1/2} \\ &\leq A'\mu(B_0)^{1/2}\|f\|_{\mathrm{bmo}}, \end{aligned} \tag{5.7}$$

where again constant A' depends only on A_2, C_D and κ. From the boundedness of the maximal function on L^∞ we get

$$\|M[f_{\widehat{B_0}}\chi_{\widehat{B_0}}]\|_\infty \le f_{\widehat{B_0}} \le F_{B_0}. \tag{5.8}$$

The last inequality was obtained by observing that for every $x \in B_0$, $f_{\widehat{B_0}} \le Mf(x) = F(x)$, and averaging over B_0. Integrating over Ω_1, using Cauchy-Schwarz and applying (5.7) and (5.8), we get

$$\begin{aligned}\int_{\Omega_1} M_1 f(x) d\mu &\le \|M[(f - f_{\widehat{B_0}})\chi_{\widehat{B_0}}]\|_2\, \mu(B_0)^{1/2} + \|M[f_{\widehat{B_0}}\chi_{\widehat{B_0}}]\|_\infty\, \mu(\Omega_1)\\ &\le C\|f\|_{\mathrm{bmo}}\mu(B_0) + F_{B_0}\mu(\Omega_1),\end{aligned}$$

which is (5.6) for $i = 1$.

For $i = 2$, the proof is identical to that in [6]. Namely, if B is such that $r(B) \ge r(B_0)$ and $B \cap B_0 \ne \emptyset$, then $\widehat{B} \supset B_0$, so as above, for every $x \in B_0$, $F(x) \ge f_{\widehat{B}}$ and therefore $F_{B_0} \ge f_{\widehat{B}}$. Thus

$$f_B - F_{B_0} \le f_B - f_{\widehat{B}} \le C_D^{N_1} \fint_{\widehat{B}} |f - f_{\widehat{B}}| d\mu \le C_D^{N_1}\|f\|_{\mathrm{bmo}}.$$

Taking the supremum over such B we get

$$M_2 f(x) - F_{B_0} \le C_D^{N_1}\|f\|_{\mathrm{bmo}}, \quad x \in B_0,$$

which when integrated over Ω_2 gives (5.6) for $i = 2$. This completes the proof of Theorem 5.1. □

§ 6. An alternative characterization of bmo

In most of the discussion above, we had to distinguish between two cases based on the size of a ball relative to the constant R chosen in Definition 3.1. Now we provide an alternative definition which does not make that distinction, and show it is equivalent to the original one.

Lemma 6.1. *Let $1 \le p < \infty$. A locally integrable function f belongs to* bmo(X) *if and only if for every ball B in X there exists a constant c_B such that*

(i)

$$M_1 = \sup_B \left[\frac{1}{\mu(B)}\int_B |f - c_B|^p d\mu\right]^{1/p} < \infty;$$

and

(ii)

$$M_2 = \sup_B \frac{|c_B|}{\log(2 + R/r(B))} < \infty.$$

Here R is the same as the constant in Definition 3.1. Furthermore,

$$\|f\|_{\mathrm{bmo}} \approx \inf \max(M_1, M_2),$$

with constants depending on p, where the infimum is taken over all choices of the $\{c_B\}$ so that (i) *and* (ii) *hold.*

In [18], the inequality in (ii) involved $\log(1 + R/r(B))$ instead of $\log(2 + R/r(B))$, which forced $c_B \to 0$ as $r(B) \to \infty$, but, as pointed out by the referee, it is only necessary for c_B to remain bounded.

When $R = \infty$, condition (ii) is null and we get an equivalent characterization of BMO (see [30], Chapter IV, Section 6.6). If we assume, as stated following Definition 3.1, that $0 < R < \operatorname{diam}(X)$ and there are balls with radii exceeding R, then the Lemma implies that the space $\operatorname{bmo}(X)$ is independent of the choice of R. As in [18], this follows from the inequality

$$\sup_B \frac{|c_B|}{\log(2 + R'/r(B))} \leq C_{R,R'} \sup_B \frac{|c_B|}{\log(2 + R/r(B))},$$

with R, R' positive real numbers and

$$C_{R,R'} = \sup_{x\in(0,\infty)} \frac{\log(2 + Rx)}{\log(2 + R'x)} < \infty. \tag{6.1}$$

Proof. Suppose $f \in \operatorname{bmo}(X)$. Then we can let $c_B = f_B$ if $r(B) < R$, $c_B = 0$ if $r(B) \geq R$. By Corollary 3.3, condition (i) is satisfied with $M_1 = \|f\|_{\mathrm{bmop}}$, which is comparable to $\|f\|_{\mathrm{bmo}}$. In addition, (ii) trivially holds for $r(B) \geq R$.

Let $B_0 = B(x_0, r_0)$ be a ball with radius $r_0 < R$, choose k to be the smallest integer such that $2^k r_0 \geq R$, and consider the chain of balls $B_i = B(x_0, 2^i r_0)$, $i = 1, \dots k$. Then

$$\begin{aligned}
|f_{B_0}| &\leq \sum_{i=1}^k |f_{B_{i-1}} - f_{B_i}| + |f_{B_k}| \\
&\leq \sum_{i=1}^k \frac{1}{\mu(B_{i-1})} \int_{B_{i-1}} |f - f_{B_i}| d\mu + \frac{1}{\mu(B_k)} \int_{B_k} |f| d\mu \\
&\leq \sum_{i=1}^k \frac{C_D}{\mu(B_i)} \int_{B_i} |f - f_{B_i}| d\mu + \frac{1}{\mu(B_k)} \int_{B_k} |f| d\mu \\
&\leq \|f\|_{\mathrm{bmo}} (C_D k + 1),
\end{aligned}$$

where C_D is the doubling constant. Note that $k \leq \log_2 \frac{R}{r_0} + 1 \leq C \log(2 + \frac{R}{r_0})$ for some numerical constant C (independent of R). Thus M_2 is bounded by a constant multiple (depending on C_D) of $\|f\|_{\mathrm{bmo}}$. It is important to point out that while we only applied the doubling condition to balls of radius smaller than R, we used the existence of a ball of radius at least as large as R in order to obtain the bound.

Conversely, suppose f satisfies conditions (i) and (ii). By Hölder's inequality we can assume the weakest condition in (i), namely $p = 1$. Then for any ball B with radius smaller than R we have

$$\frac{1}{\mu(B)} \int_B |f - f_B| d\mu \leq \frac{1}{\mu(B)} \int_B |f - c_B| d\mu + |f_B - c_B| \leq \frac{2}{\mu(B)} \int_B |f - c_B| \leq 2M_1,$$

while if $r(B) \geq R$ we have

$$\frac{1}{\mu(B)} \int_B |f| d\mu \leq \frac{1}{\mu(B)} \int_B |f - c_B| d\mu + |c_B| \leq M_1 + M_2 \log 3.$$

This shows $f \in \operatorname{bmo}(X)$ with norm bounded by a constant multiple of $\max(M_1, M_2)$. □

§ 7. The space h^1: atomic decomposition and duality

The space $\mathrm{bmo}(\mathbb{R}^n)$ was shown by Goldberg [20] to be the dual of the local Hardy space $h^1(\mathbb{R}^n)$. In this section we define a version of h^1 in the context of a space of homogeneous type (X, d, μ), following the work of Macías and Segovia [24], [25]. They showed that there exists a quasi-metric (which we call ρ) equivalent to our original one which has some Hölder continuity in each variable, namely, there exists an $\alpha \in (0,1)$ and a constant C_ρ such that for all $x \in X$ and $r > 0$

$$|\rho(y,x) - \rho(z,x)| \le C_\rho r^{1-\alpha} \rho(y,z)^\alpha \tag{7.1}$$

whenever $y, z \in B(x,r)$. Here we abuse notation by writing $B(x,t)$ to denote the balls in the new quasi-metric ρ. The fact that d and ρ are equivalent means that there exist positive constants c_1, c_2 such that, for all x and y,

$$c_1 \rho(x,y) \le d(x,y) \le c_2 \rho(x,y).$$

Consequently the corresponding balls B_d and B_ρ are nested, i.e. $B_d(x, c_1 t) \subset B_\rho(x,t) \subset B_d(x, c_2 t)$, and furthermore by doubling

$$\mu(B_\rho(x,t)) \approx \mu(B_d(x,t)).$$

It is therefore natural to use the notation $B(x,t)$ to denote the balls in ρ, with the understanding that if we want to go back to d we may have to change to dilates of these balls. Note also that because the quasi-distance ρ is continuous in each variable, the balls are now open, which will allow us to use results where this assumption is made. We will denote the constant in the quasi-triangle-inequality for ρ by κ'.

Using the α above, we state the following maximal function characterization of the local Hardy space (see [25] for a related maximal function definition for h^p distributions on spaces of homogeneous type):

Definition 7.1. Fix a positive real number T. A locally integrable function f on X is said to belong to $h^1(X)$ if $\mathcal{M}_{\mathcal{F}}(f) \in L^1(X)$. Here

$$\mathcal{M}_{\mathcal{F}}(f)(x) := \sup_{\psi \in \mathcal{F}_x} \left| \int f \psi \, d\mu \right|,$$

where $\psi \in \mathcal{F}_x$ means ψ is a α-Hölder continuous function supported in a ball $B(x,t)$, $0 < t < T$, with

$$\|\psi\|_\infty \le \frac{C_{\mathcal{F}}}{\mu(B(x,t))}, \quad \|\psi\|_{C^\alpha} \le \frac{C_{\mathcal{F}}}{t^\alpha \mu(B(x,t))}. \tag{7.2}$$

The constant $C_{\mathcal{F}}$ will be specified below. We set $\|f\|_{h^1} := \|\mathcal{M}_{\mathcal{F}}(f)\|_1$.

It follows from the definition that we can bound $\mathcal{M}_{\mathcal{F}}$ by the Hardy-Littlewood maximal function:

$$\mathcal{M}_{\mathcal{F}}(f)(x) \le \sup_{0<t<T} \frac{C_{\mathcal{F}}}{\mu(B(x,t))} \int_{B(x,t)} |f| \, d\mu \le C_{\mathcal{F}} M(f)(x).$$

However, in order to be able to show that f itself is controlled by $\mathcal{M}_{\mathcal{F}}$, we need to be able to construct an appropriate α-Hölder approximation to the identity $\{\varphi_t^x\}_{t>0} \subset \mathcal{F}_x$ at each $x \in X$. That is, we want to be able to write, for $f \in L^1(X)$, that

$$|f(x)| = \lim_{t \to 0} \left| \int f \varphi_t^x \, d\mu \right| \le \mathcal{M}_{\mathcal{F}}(f)(x) \quad \text{for a.e. } x \in X. \tag{7.3}$$

We start with a Lipschitz-continuous function ζ on $\mathbb{R}$ with $\|\zeta\|_\infty = 1$, $\|\zeta\|_{\mathrm{Lip}} = 2$, which is equal to 1 for $x \leq 1/2$ and to 0 for $x \geq 1$. Let

$$\zeta_t^x(y) = \zeta\Big(\frac{\rho(x,y)}{t}\Big). \tag{7.4}$$

Then

$$\chi_{B(x,t/2)} \leq \zeta_t^x \leq \chi_{B(x,t)}$$

so if we set $\varphi_t^x = \frac{\zeta_t^x}{\int \zeta_t^x d\mu}$ we get

$$\frac{\chi_{B(x,t/2)}}{\mu(B(x,t))} \leq \varphi_t^x \leq \frac{\chi_{B(x,t)}}{\mu(B(x,t/2))}.$$

By doubling, this means $\|\varphi_t^x\|_\infty$ is controlled by $\frac{C_D}{\mu(B(x,t))}$.

To check the Hölder continuity, fix x, t and take y, z in X. We can assume that at least one of the two points lies in the support of φ_t^x, so suppose $y \in B(x,t)$. If $\rho(y,z) \leq t$ then $\rho(x,z) \leq 2\kappa' t$ so using (7.1) with $r = 2\kappa' t$, we have

$$|\varphi_t^x(y) - \varphi_t^x(z)| \leq \frac{\|\zeta\|_{\mathrm{Lip}}|\rho(x,y) - \rho(x,z)|}{t\int \zeta_t^x d\mu} \leq \frac{2C_D C_\rho (2\kappa')^{1-\alpha}}{t^\alpha \mu(B(x,t))}\rho(y,z)^\alpha. \tag{7.5}$$

If $\rho(y,z) > t$ then

$$|\varphi_t^x(y) - \varphi_t^x(z)| \leq 2\|\varphi_t^x\|_\infty \leq \frac{2C_D}{\mu(B(x,t))} \leq \frac{2C_D}{t^\alpha \mu(B(x,t))}\rho(y,z)^\alpha. \tag{7.6}$$

Thus φ_t^x satisfies (7.2) if, say, we choose $C_{\mathcal{F}} \geq 4C_D C_\rho \kappa'$ (we may assume $C_\rho \geq 1$).

This shows that the functions φ_t^x belong to $\mathcal{F}_x$, so the averages $\int f\varphi_t^x d\mu$ are controlled by $\mathcal{M}_{\mathcal{F}}(f)(x)$, and hence by $Mf(x)$. Moreover, because $\int \varphi_t^x d\mu = 1$, and the balls in the quasimetric ρ also form a local base for the topology, the equality on the left-hand-side of (7.3) holds for every function f which is continuous at x. The result for integrable f follows the weak-type bound on the Hardy-Littlewood maximal function, as in the proof of the Lebesgue differentiation theorem, if we assume (as, for example, in [13]) that the continuous functions with bounded support are dense in $L^1(\mu)$.

The proof that h^1 is complete (see [18]), showing that every absolutely convergent series converges in the h^1 norm, applies in this setting as well.

The choice of the constant T affects the norm in Definition 7.1. In the following two definitions, we will use the constant R from Definition 3.1. The relation between T and R will become clear in Proposition 7.5 and Theorem 7.7 (ii).

Definition 7.2. Let $1 < q \leq \infty$. We say a function a is a $(1,q)$-atom if a is supported in a ball B for which the following hold:

(i)

$$\|a\|_q \leq \frac{1}{\mu(B)^{1/p}}, \quad \frac{1}{p} + \frac{1}{q} = 1;$$

and

(ii) if $r(B) < R$ then

$$\int a d\mu = 0.$$

The definition of an atom a implicitly associates to it a ball B, but there are many possible choices of B. For example, one can always choose a ball B containing the support of a and having $r(B) \geq R$, which would mean that the cancellation condition in (ii) does not apply to a. However, if this causes the measure of the ball to increase, the size condition in (i) becomes more restrictive. The following alternative definition of atoms, analogous to the alternative characterization of bmo in Lemma 6.1, includes a cancellation condition which is not conditional on whether the radius of the supporting ball is smaller or larger than R.

Definition 7.3. Let $1 < q \leq \infty$. We say a function a is an *approximate* $(1, q)$-atom if a is supported in a ball B for which the following hold:

(i)

$$\|a\|_q \leq \frac{1}{\mu(B)^{1/p}}, \quad \frac{1}{p} + \frac{1}{q} = 1;$$

and

(ii)

$$\left| \int a d\mu \right| \leq \frac{2}{\log(2 + R/r(B))}.$$

Remarks 7.4.

1. *If a is a $(1, q)$ atom as in Definition 7.2, then a is an approximate $(1, q)$-atom: the size condition (i) is the same in both definitions, and as for cancellation, if the ball B containing the support of a has radius $r(B) < R$ then $\int a d\mu = 0$, while if $r(B) \geq R$ then by the size condition*
$$\left| \int a d\mu \right| \leq \|a\|_q \mu(B)^{1/p} \leq 1 \leq \frac{2}{\log(2 + R/r(B))}.$$
2. *By Lemma 6.1, the pairing of an approximate $(1, q)$-atom a with a function $b \in \mathrm{bmo}$ can be bounded as follows (assuming B is the ball containing the support of a):*
$$\begin{aligned}\left| \int a b d\mu \right| &\leq \left| \int_B a(b - c_B) d\mu \right| + |c_B| \left| \int a d\mu \right| \\ &\leq \frac{1}{\mu(B)^{1/p}} \left[\int_B |b - c_B|^p d\mu \right]^{1/p} + \frac{2|c_B|}{\log(2 + R/r(B))} \leq C\|b\|_{\mathrm{bmo}}.\end{aligned}$$
3. *With regards to the choice of R in Definition 7.3, an approximate $(1, q)$-atom a for some R will be a multiple of an approximate $(1, q)$-atom for any other (positive, finite) R', since instead of condition (ii), a will satisfy*
$$\left| \int a d\mu \right| \leq \frac{2C_{R',R}}{\log(2 + R'/r(B))},$$
where $C_{R',R}$ is the constant in (6.1) with R, R' reversed.

While an approximate $(1, q)$-atom does not satisfy Definition 7.2, the following two results will show that it lies in h^1 and therefore can be decomposed into $(1, q)$-atoms.

Proposition 7.5. *Let $1 < q \leq \infty$ and assume $T = 4R$, where T and R are the constants in Definition 7.1 and Definition 7.2, respectively. There exists a constant $C_q < \infty$, depending on q and the constants C_D, $C_{\mathcal{F}}$, κ' and α, such that if a is an approximate $(1, q)$-atom, then*

$$\|a\|_{h^1} \leq C_q \quad \textit{with} \quad C_q = \mathcal{O}\Big(\frac{q}{q-1}\Big) \quad \textit{as} \quad q \to 1. \tag{7.7}$$

By Remark 7.4.1, the result applies a fortiori to atoms satisfying Definition 7.2.

Proof. Let a be an approximate $(1,q)$-atom supported in a ball $B_0 = B(y_0, r_0)$. We want to show that the maximal function $\mathcal{M}_{\mathcal{F}}(a)$, as defined in Definition 7.1, is in L^1 with norm bounded by C_q. If $x \in X$, as noted following Definition 7.1, we can bound $\mathcal{M}_{\mathcal{F}}$ by the Hardy-Littlewood maximal function: $\mathcal{M}_{\mathcal{F}}(a)(x) \le C_{\mathcal{F}} M(a)(x)$.

Denote by m_1 be the smallest integer with $2^{m_1} \ge \kappa' + 1$. By the L^q boundedness of the maximal function and the size condition (i),

$$\int_{(1+\kappa')B_0} \mathcal{M}_{\mathcal{F}}(a) d\mu \le \|\mathcal{M}_{\mathcal{F}}(a)\|_q \, \mu((1+\kappa')B_0)^{1/p} \le A_q \|a\|_q \, \mu(2^{m_1} B_0)^{1/p} \le A_q C_D^{m_1/p},$$

where we recall that the constant A_q in the bound for the maximal function satisfies $A_q = \mathcal{O}\big(\frac{q}{q-1}\big)$ as $q \to 1$ (see [30], I.3.1). Note that we did not use the approximate cancellation condition (ii) for this part.

Now fix $x \notin (1+\kappa')B_0$, and let $\psi \in \mathcal{F}_x$, namely ψ is supported in a ball $B(x,t)$ and satisfies the Hölder bounds in (7.2). For a we will use the cancellation condition (ii) from Definition 7.3, and we will only need a weaker size condition, namely (i) with $q = 1$: $\|a\|_1 \le 1$. Thus the constants involved will be independent of q. Write

$$\begin{aligned} \left| \int a\psi d\mu \right| &\le \int_{B_0} |a(y)||\psi(y) - \psi(y_0)| d\mu + |\psi(y_0)| \left| \int_{B_0} a(y) \right| \\ &\le \|\psi\|_{C^\alpha} r_0^\alpha + \frac{\|\psi\|_\infty}{\log(2 + R/r_0)} \\ &\le \frac{C_{\mathcal{F}}}{\mu(B(x,t))} \left[\left(\frac{r_0}{t} \right)^\alpha + \frac{1}{\log(2+R/r_0)} \right]. \end{aligned} \tag{7.8}$$

In order for $\int a\psi d\mu \neq 0$, there must exist $y \in B_0 \cap B(x,t)$. Since $\rho(y, y_0) \le r_0 \le \frac{\rho(x,y_0)}{1+\kappa'}$, we have

$$t \ge \rho(x,y) \ge \frac{\rho(x,y_0)}{\kappa'} - \rho(y,y_0) \ge \frac{\rho(x,y_0)}{\kappa'} - \frac{\rho(x,y_0)}{1+\kappa'} = \frac{\rho(x,y_0)}{\kappa'(1+\kappa')} \tag{7.9}$$

Combining (7.8) and (7.9) we get the bound

$$\left| \int a\psi d\mu \right| \le \frac{C_{\mathcal{F}}}{\mu(B(x,t))} \left[\left(\frac{\kappa'(\kappa'+1) r_0}{\rho(x,y_0)} \right)^\alpha + \frac{1}{\log(2+R/r_0)} \right],$$

but in order to take the supremum we need to make this estimate independent of t.

Let $j \ge 2$ be the unique integer such that $(1+\kappa')^{j-1} r_0 \le \rho(x,y_0) < (1+\kappa')^j r_0$. Taking $z \in (\kappa'+1)^j B_0$, we have

$$\rho(z,x) \le \kappa'[\rho(z,y_0) + \rho(x,y_0)] \le \kappa'[(\kappa'+1)^j r_0 + \rho(x,y_0)] \le \kappa'(\kappa'+2)\rho(x,y_0)$$

which combined with the fact that $\rho(x,y_0) \le \kappa'(\kappa'+1)t$ by (7.9), and doubling, gives

$$\mu((\kappa'+1)^j B_0) \le C_D^{m_2} \mu(B(x,t)),$$

where m_2 is the smallest integer with $2^{m_2} \ge (\kappa')^2(\kappa'+1)(\kappa'+2)$. Thus we get

$$\left| \int a\psi d\mu \right| \le \frac{C_D^{m_2} C_{\mathcal{F}}}{\mu((\kappa'+1)^j B_0)} \left[\left(\kappa'(1+\kappa')^{2-j} \right)^\alpha + \frac{1}{\log(2+R/r_0)} \right]. \tag{7.10}$$

Taking the supremum on the left over $\psi \in \mathcal{F}_x$, we see that the same estimate holds for $\mathcal{M}_{\mathcal{F}}(a)(x)$.

Now we integrate in x over $X \setminus (1+\kappa')B_0$. Since in (7.9) we have $t < T$, we see that we need only consider $x \in B(y_0, (1+\kappa')T)$. As in the proof of Lemma 6.1, let k be the smallest integer such that $(1+\kappa')^k r_0 \geq (1+\kappa')T$ and take a chain of balls $B_j = (1+\kappa')^j B_0$, $j = 1, \dots, k$. By (7.10),

$$\begin{aligned}\int_{X\setminus(1+\kappa')B_0} \mathcal{M}_{\mathcal{F}}(a)d\mu &= \sum_{j=2}^{k} \int_{B_j\setminus B_{j-1}} \mathcal{M}_{\mathcal{F}}(a)d\mu \\ &\leq C_D^{m_2} C_{\mathcal{F}}(\kappa')^\alpha \sum_{j=2}^{k} \left[(1+\kappa')^{\alpha(2-j)} + \frac{1}{\log(2+R/r_0)}\right] \\ &\leq C_D^{m_2} C_{\mathcal{F}}(\kappa')^\alpha \left[\sum_{i=0}^{\infty}(1+\kappa')^{-\alpha i} + \frac{k}{\log(2+R/r_0)}\right] \\ &\leq C,\end{aligned}$$

where C depends on C_D, $C_{\mathcal{F}}$, κ' and α. In the last step, as in the proof of Lemma 6.1, we used the fact that $k \leq \log_{(1+\kappa')}\left(\frac{T}{r_0}\right) + 2 \leq C\log(2+\frac{R}{r_0})$ since $T = 4R$. $\square$

Proposition 7.5 and the completeness of h^1 imply that if $\{\lambda_j\}$ is a sequence in ℓ^1 and $\{a_j\}$ is a sequence of (approximate) $(1,q)$-atoms, then $\sum \lambda_j a_j$ converges to a function in h^1. The converse is contained in the following theorem.

Theorem 7.6. *If $f \in h^1(X)$, then there exists a sequence of $(1,\infty)$ atoms $\{a_j\}$ and a sequence of coefficients $\{\lambda_j\} \in \ell^1$ such that*

$$f = \sum \lambda_j a_j$$

and

$$\sum |\lambda_j| \leq C\|f\|_{h^1}.$$

In order to prove Theorem 7.6, we need the following version of the Calderón-Zygmund decomposition. In this version, instead of $T = 4R$ as above, we will use $T = 4(\kappa')^2 R$. When we apply this to prove the atomic decomposition, it will results in atoms satisfying Definition 7.2 with a value of R which is $(\kappa')^{-2}$ times the old value of R. These new atoms, by Remark 7.4.1 and Remark 7.4.3, will be multiples of approximate $(1,\infty)$-atoms with respect to the old constant R, thus giving the desired atomic decomposition. Conversely, by Proposition 7.5, any function decomposed in terms of atoms relative to the new R will still be in $h^1(X)$, but the norm will change by a constant factor, showing the space is invariant under a change in the constant R (as long as there exist balls B with $r(B) \geq R$).

Theorem 7.7. *Given $f \in L^1_{\mathrm{loc}}(X)$, $\alpha > 0$ and $C_0 > 4\kappa'$, we can write*

$$f = g + b, \quad b = \sum_{k=1}^{\infty} b_k$$

for some functions g, b_k, and a sequence of balls $\{B_k\}_{k=1}^{\infty}$ satisfying

(i) *$\|g\|_\infty \leq c\alpha$ for some $c \geq 1$ depending on C_0, κ', α, C_ρ and C_D;*

(ii) $\operatorname{supp}(b_k) \subset B_k^* := C_0 B_k$, *and*

$$\int b_k d\mu = 0 \quad \text{when } r(B_k^*) < R = \frac{T}{4(\kappa')^2};$$

(iii)

$$\|b_k\|_1 \le 2c \int_{B_k^*} \mathcal{M}_{\mathcal{F}} f;$$

and

(iv) *the balls* B_k^* *have bounded overlap and*

$$\bigcup B_k^* = \{x \in X : \mathcal{M}_{\mathcal{F}} f(x) > \alpha\}.$$

Proof. Let $U_\alpha = \{x \in X : \mathcal{M}_{\mathcal{F}}(x) > \alpha\}$, $F_\alpha = X \setminus U_\alpha$. We use the Whitney-type covering lemma for U_α proved by Coifman and Weiss for spaces of homogeneous type (see [15], Theorem 3.2, with $C = C_0$), which is valid here based on the doubling assumption and the fact that the balls in the quasi-metric ρ are open, to get the sequence of balls $\{B_k\}$ satisfying

(1) $U_\alpha = \cup B_k = \cup B_k^*$ with $B_k^* = C_0 B_k$;

(2) the balls B_k^* have bounded overlap, and

(3) $3\kappa' B_k^* \cap F_\alpha \neq \emptyset$.

Note that (iv) is just a restatement of properties (1) and (2).

We now take a partition of unity $\{\eta_k\}_k$ subordinate to the cover $\{\widetilde{B_k}\}_k$. Here we change a bit the dilation factor and set $\widetilde{B_k}$ to be $K' B_k$, where $2 < K' = \frac{C_0}{2\kappa'} < C_0$, so that $2B_k \subset \widetilde{B_k} \subset B_k^*$. To define the partition of unity, let ζ be the Lipschitz function with norm 2 which is used in the definition of the approximation to the identity above. For each k, denote by x_k the center of B_k and r_k its radius and define, as in (7.4) (with x replaced by x_k and t replaced by $2r_k$),

$$\zeta_k(y) := \zeta\Big(\frac{\rho(x_k, y)}{2r_k}\Big).$$

Proceeding as in the proof of the estimates (7.5), (7.6), and using (7.1), we have that

$$\|\zeta_k\|_{C^\alpha} \le \max\Big(\frac{\|\zeta\|_{\mathrm{Lip}} C_\rho (4\kappa' r_k)^{1-\alpha}}{2r_k}, \frac{2\|\zeta\|_\infty}{(2r_k)^\alpha}\Big) \le C' r_k^{-\alpha},$$

where C' depends on C_ρ, κ' and α. Note also that on U_α,

$$1 \le \sum \zeta_k \le \sum \chi_{\widetilde{B_k}} \le M < \infty$$

by properties (1) and (2). Moreover, if $x \in \widetilde{B_j} \cap \widetilde{B_k}$, and say $r_j \le r_k$, then using properties (1) and (3), taking $y \in 3\kappa' B_j^* \cap F_\alpha$, we can write

$$\begin{aligned} C_0 r_k &\le \rho(x_k, y) \le \kappa'[\rho(x_k, x) + \rho(x, y)] \\ C_0 r_k &\le \kappa' \frac{C_0}{2\kappa'} r_k + (\kappa')^2[\rho(x, x_j) + \rho(x_j, y)] \\ (C_0 - \frac{C_0}{2}) r_k &\le \kappa' \frac{C_0}{2} r_j + (\kappa')^2 \rho(x_j, y) \\ \frac{C_0}{2} r_k &\le \kappa'(\frac{C_0}{2} + 3(\kappa')^2 C_0) r_j \\ \frac{r_k}{r_j} &\le \widetilde{C} := \kappa'(1 + 6(\kappa')^2). \end{aligned} \tag{7.11}$$

Thus r_j, r_k are comparable whenever $\widetilde{B_j} \cap \widetilde{B_k} \neq \emptyset$. Letting $\eta_k = \frac{\zeta_k}{\sum \zeta_k}$, we get our desired partition of unity, with

$$\frac{1}{M}\chi_{B_k} \leq \eta_k \leq \chi_{\widetilde{B_k}} \tag{7.12}$$

and

$$\|\eta_k\|_{C^\alpha} \leq \|\zeta_k\|_{C^\alpha} + \sum_{\widetilde{B_j} \cap \widetilde{B_k} \neq \emptyset} \|\zeta_j\|_{C^\alpha} \leq C'[r_k^{-\alpha} + \sum_{\widetilde{B_j} \cap \widetilde{B_k} \neq \emptyset} r_j^{-\alpha}] \leq C'(1 + M\widetilde{C}^\alpha) r_k^{-\alpha}. \tag{7.13}$$

Now set

$$b_k = (f - c_k)\eta_k, \tag{7.14}$$

where $c_k = 0$ if $r_k \geq R/C_0$, and $c_k = \frac{\int f\eta_k}{\int \eta_k}$ if $r_k < R/C_0$. This gives us property (ii). Letting $b = \sum b_k$, we see that b is supported in U_α and

$$g = f - b = f\chi_{F_\alpha} + \sum c_k \eta_k.$$

Then for almost every x, by (7.3) and the definition of F_α,

$$|g(x)| \leq |f(x)|\chi_{F_\alpha}(x) + \sum |c_k|\eta_k(x) \leq \mathcal{M}_{\mathcal{F}}(x)\chi_{F_\alpha}(x) + \sum |c_k|\eta_k(x) \leq c\alpha, \tag{7.15}$$

provided we can show that $|c_k| \leq c\alpha$ for every k. This is certainly true if $r_k \geq R/C_0$, so assume $r_k < R/C_0$. By property (3), there exists $y \in F_\alpha$ with $\rho(y, x_k) \leq 3\kappa C_0 r_k$, so by the quasi-triangle inequality, if $z \in \widetilde{B_k}$ then

$$\rho(z, y) \leq \kappa'[\rho(z, x_k) + \rho(x_k, y)] \leq \kappa'[\frac{C_0}{2\kappa'} + 3\kappa' C_0] r_k = [\frac{1}{2} + 3(\kappa')^2] C_0 r_k.$$

Thus η_k vanishes outside $B(y, t_k)$, where $t_k := [\frac{1}{2} + 3(\kappa')^2]C_0 r_k < 4(\kappa')^2 R = T$ and, again by the quasi-triangle inequality, $B(y, t_k) \subset B(x_k, 7(\kappa')^3 C_0 r_k)$, so from doubling we get

$$\mu(B(y, t_k)) \leq C_1 \mu(B_k) \tag{7.16}$$

for a constant $C_1 \geq 1$ depending on C_0, κ' and the doubling constant C_D. So it remains to show that η_k is a constant multiple of a function in $\mathcal{F}_y$. We let $\widetilde{\eta_k} = \frac{\eta_k}{c \int \eta_k}$, where c will be chosen appropriately. Using (7.12) and (7.16),

$$\|\widetilde{\eta_k}\|_\infty \leq \|\eta_k\|_\infty M(c\mu(B_k))^{-1} \leq MC_1(c\mu(B(y, t_k)))^{-1} \leq C_{\mathcal{F}}\mu(B(y, t_k))^{-1}$$

provided we take $c \geq MC_1 C_{\mathcal{F}}{}^{-1}$, and by (7.12) and (7.13),

$$\|\widetilde{\eta_k}\|_{C^\alpha} \leq C'(1 + M\widetilde{C}^\alpha) M r_k^{-\alpha}(c\mu(B_k))^{-1} \leq C_{\mathcal{F}} t_k^{-\alpha} \mu(B(y, t_k)))^{-1}$$

provided $c \geq C'(1 + M\widetilde{C}^\alpha) MC_1([\frac{1}{2} + 3(\kappa')^2]C_0)^\alpha$ (we may assume $C_{\mathcal{F}} \geq 1$). Hence by a suitable choice of c we get $\widetilde{\eta_k} \in \mathcal{F}_y$, so that by Definition 7.1,

$$|c_k| = c\left|\int f\widetilde{\eta_k} d\mu\right| \leq c\mathcal{M}_{\mathcal{F}}(y) \leq c\alpha. \tag{7.17}$$

This proves (i).

Now let us prove (iii). By (ii), (7.3), (7.17), and the fact that $B_k^* \subset U_\alpha$, we have

$$\begin{aligned}\|b_k\|_1 = \int_{B_k^*} |b_k| d\mu &\leq \int_{B_k^*} |f\eta_k| d\mu + |c_k| \|\eta_k\|_\infty \mu(B_k^*)\\ &\leq \int_{B_k^*} \mathcal{M}_{\mathcal{F}}(f\eta_k) d\mu + c\alpha\mu(B_k^*)\\ &\leq \int_{B_k^*} \mathcal{M}_{\mathcal{F}}(f\eta_k) d\mu + c\int_{B_k^*} \mathcal{M}_{\mathcal{F}}(f) d\mu.\end{aligned}$$

Thus it remains to show that

$$\mathcal{M}_{\mathcal{F}}(f\eta_k)(x) \leq c\mathcal{M}_{\mathcal{F}}(f)(x), \quad x \in B_k^* \tag{7.18}$$

Fix $x \in B_k^*$ and take $\psi \in \mathcal{F}_x$ with support in $B(x,t)$ for some $t < T$. Consider the C^α function $\eta_k\psi$, which satisfies

$$\|\eta_k\psi\|_\infty \leq \|\psi\|_\infty \leq C_{\mathcal{F}}\mu(B(x,t))^{-1} \tag{7.19}$$

and

$$\|\eta_k\psi\|_{C^\alpha} \leq \|\eta_k\|_{C^\alpha}\|\psi\|_\infty + \|\psi\|_{C^\alpha} \leq C_{\mathcal{F}}[C'(1+M\widetilde{C}^\alpha)r_k^{-\alpha} + t^{-\alpha}]\mu(B(x,t))^{-1}. \tag{7.20}$$

We will show $\varphi = c^{-1}\eta_k\psi \in \mathcal{F}_x$, for a choice of c consistent with the one above, by showing that this function satisfies the conditions (7.2) with respect to the ball $B(x,s)$, with $s = \min(t, [\frac{1}{2}+\kappa']C_0 r_k) < T$. Since $x \in B_k^*$, if $y \in \operatorname{supp}(\varphi) \subset \widetilde{B_k} \cap B(x,t)$ then

$$\rho(x,y) \leq \kappa'[\rho(x,x_k) + \rho(x_k,y)] \leq \kappa'(C_0 + \frac{C_0}{2\kappa'})r_k = C_0(\kappa' + \frac{1}{2})r_k,$$

and $\rho(x,y) < t$, so $y \in B(x,s)$. From (7.19), since $s \leq t$, we have $\|\eta_k\psi\|_\infty \leq C_{\mathcal{F}}\mu(B(x,s))^{-1}$. Recall that we had previously required $c \geq C'(1+M\widetilde{C}^\alpha)MC_1([\frac{1}{2}+3(\kappa')^2]C_0)^\alpha$, so adding 1 to this choice and using the fact that all the constants M, C_1 and κ' are at least one, we can bound the right-hand-side of (7.20) by

$$C_{\mathcal{F}}\Big\{C'(1+M\widetilde{C}^\alpha)[\frac{1}{2}+\kappa']^\alpha C_0^\alpha + 1\Big\}s^{-\alpha}\mu(B(x,s))^{-1} \leq cC_{\mathcal{F}}s^{-\alpha}\mu(B(x,s)^{-1},$$

showing that $c^{-1}\eta_k\psi \in \mathcal{F}_x$ and therefore

$$\left|\int f\eta_k\psi d\mu\right| \leq c\mathcal{M}_{\mathcal{F}}(f)(x).$$

Taking the supremum on the left over all such ψ gives (7.18), which completes the proof. □

We are now ready to prove Theorem 7.6.

Proof. Take $f \in h^1(X)$. For each integer j, apply the Calderón-Zygmund decomposition corresponding to the set $U^j = \{\mathcal{M}_{\mathcal{F}}(f)(x) > 2^j\}$ (i.e. Theorem 7.7 with $\alpha = 2^j$), with F^j denoting the complement of U^j. This gives us functions g^j and $b^j = \sum b_k^j$ with $f = g^j + b^j$. We want to write

$$f = \sum_{-\infty}^{\infty}(g^{j+1} - g^j) = \sum_{-\infty}^{\infty}(b^j - b^{j+1}). \tag{7.21}$$

Let us check that this sum converges in L^1, namely $g^j \to f$ as $j \to \infty$ and $g^j \to 0$ as $j \to -\infty$. The former follows from properties (iii) and (iv) of the Calderón-Zygmund decomposition by writing:

$$\|f - g^j\|_1 = \|b^j\|_1 \leq \sum_k \|b^j_k\|_1 \leq 2c \sum_k \int_{(B^j_k)^*} \mathcal{M}_{\mathcal{F}}(f) d\mu \leq 2Mc \int_{\{\mathcal{M}_{\mathcal{F}}(f) > 2^j\}} \mathcal{M}_{\mathcal{F}}(f) d\mu,$$

where M is the maximum intersection number of the $(B^j_k)^*$, and noting that the integral on the right converges to zero as $j \to \infty$ by the integrability of $\mathcal{M}_{\mathcal{F}}(f)$. Similarly, the latter follows by writing, thanks to (7.15),

$$\|g^j\|_1 = \int_{U^j} |g^j| d\mu + \int_{F^j} |g^j| d\mu \leq c2^j \mu(\{\mathcal{M}_{\mathcal{F}}(f)(x) > 2^j\}) + \int_{\{\mathcal{M}_{\mathcal{F}}(f) \leq 2^j\}} |\mathcal{M}_{\mathcal{F}}(f)| d\mu$$

and again using the integrability of $\mathcal{M}_{\mathcal{F}}(f)$ to conclude that both terms converge to zero as $j \to -\infty$.

Since $U^j \supset U^{j+1}$ contains the support of b^{j+1}, we can use the partition of unity $\{\eta^j_k\}$ from the proof of Theorem 7.7, corresponding to the decomposition of U^j, to re-write (7.21) as

$$f = \sum_j \left[b^j - b^{j+1} \sum_k \eta^j_k \right] = \sum_j \sum_k h^j_k.$$

From the definition of b^j_k and b^{j+1}_l (i.e. (7.14) relative to the functions η^j_k and η^{j+1}_l and respective constants constants c^j_k, c^{j+1}_l), we can write each term in the sum as

$$\begin{aligned} h^j_k &:= b^j_k - \sum_l b^{j+1}_l \eta^j_k \\ &= \left[(f - c^j_k) - \sum_l (f - c^{j+1}_l) \eta^{j+1}_l \right] \eta^j_k \\ &= \left[f(1 - \sum_l \eta^{j+1}_l) - c^j_k + \sum_l c^{j+1}_l \eta^{j+1}_l \right] \eta^j_k. \end{aligned} \tag{7.22}$$

Since $1 - \sum_l \eta^{j+1}_l = \chi_{F^{j+1}}$,

$$\|h^j_k\|_\infty \leq \left[2^{j+1} + c2^j + c2^{j+1} \sum_l \eta^{j+1}_l \right] \|\eta^j_k\|_\infty \leq c' 2^j, \tag{7.23}$$

where we have used (7.3) and (7.17) with $\alpha = 2^j$ and $\alpha = 2^{j+1}$.

We will build our atoms from these functions, but we need to make sure that they have vanishing integral if their support has radius smaller than R. For that we may need to add some terms to h^j_k. Let L^j_k denote the set of indices l which correspond to the nonzero terms of the sum in (7.22), namely those for which $\operatorname{supp}(\eta^{j+1}_l) \cap \operatorname{supp}(\eta^j_k) \neq \emptyset$. Recalling the construction of the partition of unity, this means for each $l \in L^j_k$, there exists $x \in \widetilde{B^{j+1}_l} \cap \widetilde{B^j_k}$. Since $F^{j+1} \supset F^j$, if $y \in F^j \cap 3\kappa'(B^j_k)^*$ and x^{j+1}_l denotes the center of B^{j+1}_l, we have

$$r((B^{j+1}_l)^*) \leq \rho(x^{j+1}_l, y) \leq \kappa'[\rho(x^{j+1}_l, x) + \rho(x, y)].$$

Continuing as in (7.11), we get that $r(B_l^{j+1}) \le \widetilde{C} r(B_k^j)$ and $(B_l^{j+1})^*$ is contained in a ball $(B_k^j)'$ which is a dilate of B_k^j of radius

$$r((B_k^j)') := \frac{C_0\kappa'}{2}(3\widetilde{C}+1)r(B_k^j) \ge \widetilde{C}C_0 r(B_k^j) = \widetilde{C} r((B_k^j)^*) \ge r((B_l^{j+1})^*). \tag{7.24}$$

For each $l \in L_k^j$, we set

$$c_{k,l}^j = \begin{cases} \frac{\int b_l^{j+1}\eta_k^j d\mu}{\int \eta_l^{j+1} d\mu} & \text{if } r((B_l^{j+1})^*) < R, \\ 0 & \text{if } r((B_l^{j+1})^*) \ge R. \end{cases} \tag{7.25}$$

Let

$$\widetilde{h_k^j} := b_k^j - \sum_{l \in L_k^j} b_l^{j+1}\eta_k^j + \sum_{l \in L_k^j} c_{k,l}^j \eta_l^{j+1}. \tag{7.26}$$

Note that $\operatorname{supp}(\widetilde{h_k^j}) \subset (B_k^j)^* \bigcup \bigcup_{l \in L_k^j} (B_l^{j+1})^* \subset (B_k^j)'$. We claim that for every j,

$$\sum_k h_k^j = \sum_k \widetilde{h_k^j}. \tag{7.27}$$

To see this, compare (7.22) and (7.26) and use (7.25) to see that the only non-zero extra terms on the right-hand-side of (7.27) form the sum

$$\sum_k \sum_{\{l \in L_k^j : r((B_l^{j+1})^*) < R\}} \frac{\int b_l^{j+1}\eta_k^j d\mu}{\int \eta_l^{j+1} d\mu}\eta_l^{j+1} = \sum_{\{l \in L_k^j : r((B_l^{j+1})^*) < R\}} \frac{\eta_l^{j+1}}{\int \eta_l^{j+1} d\mu} \int b_l^{j+1}\Big[\sum_k \eta_k^j\Big] d\mu = 0.$$

Here we have used the fact that $\sum_k \eta_k^j = 1$ on $U^j \supset U^{j+1}$, and property (ii) of the Calderón-Zygmund decomposition.

Now we check the moment condition. This need only apply if $r((B_k^j)') < R$, which by (7.24) forces $r((B_k^j)^*) < R$ and $r((B_l^{j+1})^*) < R$ for every $l \in L_k^j$, hence

$$\int \widetilde{h_k^j} d\mu = \int b_k^j d\mu - \sum_{l \in L_k^j} \int b_l^{j+1}\eta_k^j d\mu + \sum_{l \in L_k^j} \frac{\int b_l^{j+1}\eta_k^j d\mu}{\int \eta_l^{j+1} d\mu} \int \eta_l^{j+1} d\mu = 0. \tag{7.28}$$

Finally, we want to estimate the size of $\widetilde{h_k^j}$. Since we already have (7.23), we only have to bound the sum involving the $c_{k,l}^j$. Write

$$\begin{aligned}
\Big\|\sum_{l \in L_k^j} c_{k,l}^j \eta_l^{j+1}\Big\|_\infty &\le \max_{l \in L_k^j, r((B_l^{j+1})^*) < R} |c_{k,l}^j| \sum_l \eta_l^{j+1} \\
&\le \max_{l \in L_k^j, r((B_l^{j+1})^*) < R} \frac{\int (f - c_l^{j+1})\eta_l^{j+1}\eta_k^j d\mu}{\int \eta_l^{j+1} d\mu} \\
&\le \max_{l \in L_k^j, r((B_l^{j+1})^*) < R} \left| \int f \frac{\eta_l^{j+1}\eta_k^j}{\int \eta_l^{j+1} d\mu} d\mu \right| + \max_{l \in L_k^j} |c_l^{j+1}|.
\end{aligned}$$

We already know that $|c_l^{j+1}| \le c2^{j+1}$ for every l. To bound the first term on the right-hand-side, we follow the same reasoning leading up to (7.17), but with the test function $\widetilde{\eta_k} = \frac{\eta_k}{c\int \eta_k}$

replaced by $\frac{\eta_l^{j+1}\eta_k^j}{\int \eta_l^{j+1} d\mu}$. This function is supported in $\widetilde{B_l^{j+1}}$, and we know that $C_0 r(B_l^{j+1}) = r((B_l^{j+1})^*) < R$, so the same calculations show that it is supported in a ball centered at a point $y \in F^{j+1}$ of radius $t < 4(\kappa')^2 R = T$. Using (7.13) and the fact that $r(B_l^{j+1}) \lesssim r(B_k^j)$ for $l \in L_k^j$, we can continue along the same lines to show that it is a constant multiple of a function in $\mathcal{F}_y$. Thus we can bound both terms from above by a constant multiple of 2^j, giving us the desired estimate

$$\|\widetilde{h_k^j}\|_\infty \leq c' 2^j.$$

Now set $a_k^j = \frac{\widetilde{h_k^j}}{c' 2^j \mu((B_k^j)')}$, $\lambda_k^j = c' 2^j \mu((B_k^j)')$. Then a_k^j is supported in $(B_k^j)'$ and satisfies condition (i) of Definition 7.2 for a $(1,\infty)$ atom. Condition (ii) follows from the corresponding moment condition (7.28) for $\widetilde{h_k^j}$ when $r((B_k^j)') < R$. We also know that

$$\sum \lambda_k^j a_k^j = \sum_j \sum_k \widetilde{h_k^j} = \sum_j \sum_k h_k^j = f,$$

with the sum converging in L^1 by (7.21). Finally, by (7.24), doubling and property (iv) of the Calderón-Zygmund decomposition,

$$\begin{aligned}
\sum |\lambda_k^j| = c' \sum_j 2^j \sum_k \mu((B_k^j)') &\leq c' C \sum_j 2^j \sum_k \mu((B_k^j)^*) \\
&\leq c' C M \sum_j 2^j \mu(\{x \in X : \mathcal{M}_{\mathcal{F}} f(x) > 2^j\}) \\
&\leq C \int \mathcal{M}_{\mathcal{F}} f d\mu = C \|f\|_{h^1}.
\end{aligned}$$

□

As in [18], the atomic decomposition can be used to show the duality of h^1 and bmo:

Corollary 7.8. *The space* $\mathrm{bmo}(X)$ *can be identified with the dual of* $h^1(X)$*, in the sense that each* $f \in \mathrm{bmo}(X)$ *defines a bounded linear functional* Λ *on* $h^1(X)$ *with*

$$\Lambda(g) = \int f g d\mu \quad \text{for } g \text{ in a dense subset of } h^1(X), \text{ and } \|\Lambda\| \approx \|f\|_{\mathrm{bmo}}. \tag{7.29}$$

Conversely, each element Λ *in the dual of* $h^1(X)$ *can be represented by a function* $f \in \mathrm{bmo}$ *in the sense of (7.29).*

Proof. The proof in [18] follows the method in [30], Chapter IV, Section 1.2 (for H^1 and BMO), and remains the same in the setting of a space of homogeneous type, but we include it here for the sake of completeness. First, as we saw in Remark 7.4.2, if $f \in \mathrm{bmo}(X)$ then for every $(1,\infty)$ atom a, supported in a ball B, we have

$$\left| \int f a d\mu \right| \leq C \|f\|_{\mathrm{bmo}}.$$

Thus f defines a bounded linear functional on the subspace of h^1 consisting of finite linear combinations of $(1,\infty)$ atoms. By Theorem 7.6, this space is dense in h^1, hence this functional can be extended to a unique element of the dual of h^1, with norm bounded by $\|f\|_{\mathrm{bmo}}$.

Conversely, we want to show that a bounded linear functional Λ on h^1 can be represented by a function $f \in$ bmo. Take a ball B with $r(B) \geq R$, and let g be any non-zero element of $L^2(B)$. Then $a = g\|g\|^{-1}_{L^2(B)}\mu(B)^{-1/2}$ is a $(1,2)$-atom so by Proposition 7.5, $\Lambda(g) \leq C_2\|\Lambda\|\|g\|_{L^2(B)}\mu(B)^{1/2}$. Thus Λ defines a bounded linear functional on $L^2(B)$ with norm bounded by $C_2\|\Lambda\|\mu(B)^{1/2}$. By the Riesz Representation Theorem for $L^2(B)$, this means

$$\Lambda(g) = \int_B f^B g d\mu$$

for some $f^B \in L^2(B)$ with $\|f^B\|_{L^2(B)} \leq C_2\|\Lambda\|\mu(B)^{1/2}$. If $B_1 \subset B_2$ are two balls with $r(B_1) \geq R$, then for every $g \in L^2(B_1)$ we have $\Lambda(g) = \int_{B_1} f^{B_1} g d\mu = \int_{B_1} f^{B_2} g d\mu$, so f^{B_1} and f^{B_2} are identical on B_1. Thus we get a uniquely defined function $f \in L^2_{\text{loc}}(X)$ with

$$\Lambda(g) = \int f g d\mu$$

for every $g \in L^2$ with compact support. In particular this holds for any $(1,\infty)$-atom a, hence for any finite linear combination of atoms, which means, by Theorem 7.6, for g in a dense subset of $h^1(X)$. Moreover, if $r(B) \geq R$ then

$$\frac{1}{\mu(B)}\int_B |f| d\mu \leq \|f\|_{L^2(B)}\mu(B)^{-1/2} \leq C_2\|\Lambda\|. \tag{7.30}$$

If we now take a ball B with $r(B) < R$, and apply the argument above but with $g \in L^2(B)$ satisfying $\int g d\mu = 0$, i.e. $g \in L^2_0(B)$, then considering the dual space, namely $L^2(B)$ modulo constants, we get

$$\inf_{\text{constant } c} \|f - c\|_{L^2(B)} \leq C_2\|\Lambda\|\mu(B)^{1/2}.$$

Since the left-hand-side is minimized when c is the mean, we have

$$\frac{1}{\mu(B)}\int_B |f - f_B| d\mu \leq \left(\frac{1}{\mu(B)}\int_B |f - f_B|^2 d\mu\right)^{1/2} \leq C_2\|\Lambda\|. \tag{7.31}$$

From (7.30) and (7.31) we conclude that f satisfies (3.3) with norm bounded by $C_2\|\Lambda\|$. $\square$

As a consequence of Corollary 7.8, we get another proof of the John-Nirenberg inequality for bmo, (3.7). Again the argument here is the same as in [18], which in turn follows closely the one given for BMO in [30], Chapter IV, Section 1.3. We use the ideas in the proof of Corollary 7.8, but with L^2 replaced by L^q, $1 < q \leq \infty$. In particular, starting with $f \in \text{bmo}(X)$, using the properties of the corresponding bounded linear functional, Λ, on h^1, we obtain the analogues of (7.30) and (7.31) with $p = \frac{q}{q-1}$ instead of 2 and $\|\Lambda\| \leq \|f\|_{\text{bmo}}$. Thus we arrive at the conclusion of Corollary 3.3 without using the John-Nirenberg inequality. Moreover, since the constant C_q obtained in the analogues of (7.30) and (7.31) is just the bound on the h^1 norm of a $(1,q)$-atom, by (7.7) we have

$$\|f\|_{\text{bmop}} \leq C_q\|f\|_{\text{bmo}} \quad \text{with} \quad C_q = \mathcal{O}(p) \quad \text{as} \quad p \to \infty.$$

From this, using Chebychev's inequality, we get, for a given ball B and $\alpha > 0$,

$$\mu(\{x \in B : |f(x) - c_B| > \alpha\}) \leq (C_q\|f\|_{\text{bmo}})^p \alpha^{-p}\mu(B) \leq \left(\frac{Ap\|f\|_{\text{bmo}}}{\alpha}\right)^p. \tag{7.32}$$

Set $\gamma = \|f\|_{\mathrm{bmo}}$. If $p = \alpha/(2A\gamma) \geq 1$ and $c = (2A)^{-1}\log 2$, the right-hand-side of (7.32) becomes $(1/2)^p = e^{-c\alpha/\gamma}$ and (7.32) is just (3.7) with $C = 1$. If $\alpha/(2A\gamma) < 1$ then $e^{-c\alpha/\gamma} \geq 1/2$ and (3.7) (with $C = 2$) reduces to the trivial estimate

$$\mu(\{x \in B : |f(x) - c_B| > \alpha\}) \leq \mu(B) \leq 2e^{-c\alpha/\gamma}\mu(B).$$

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